A Summary of the asymptotic analysis for the EPRL amplitude

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Abstract

We review the basic steps in building the asymptotic analysis of the Euclidean sector of new spin foam models using coherent states, for Immirzi parameter less than one. We focus on conceptual issues and by so doing omit peripheral proofs and the original discussion on spin structures.

1 Introduction

The present work consists in the report of a talk given by one of us (HG) in the Planck Scale 2009 Conference, which took place in Wroclaw, and is based entirely on [1].

A spin foam model [2] is a procedure to compute an amplitude from a triangulated manifold $\mathcal{T}$ with n-simplices Δ_n coloured by representation theory data. In four-dimensions, such an amplitude is typically of the form

$$\mathcal{Z}(\mathcal{T}) = \sum_{\iota,\rho} \prod_{\Delta_2} f_2(\rho) \prod_{\Delta_3} f_3(\rho,\iota) \prod_{\Delta_4} f_4(\rho,\iota)\ , \tag{1}$$

where f_n are weights assigned to the n-simplices of the triangulated manifold, and ρ and ι respectively denote the assignments of unitary, irreducible representations to the 2-simplices, and intertwining operators to the 3-simplices of $\mathcal{T}$. The model is specified by the choice of representation assignments, the vector space of intertwining operators ι, and weights f_n.

A key step in understanding the semiclassical regime of a spin foam model in dimension d is the analysis of the asymptotic behaviour of the d-simplex amplitude that defines the model. For instance, what really established the Ponzano-Regge spin foam model as a model for 3D quantum gravity [3] was the discovery that it had some very tangible geometric interpretation, *in the asymptotic limit*. The discovery by Ponzano and Regge that it contains the geometry of the tetrahedron through the Regge action was the crucial step for the corresponding spin foam model.

Similar asymptotic analysis of the 4-dimensional models [4] was initially performed by Barrett and Williams [5], and formed the basis of investigations of the graviton propagator structure of these models [6]. This latter analysis showed a definite incompatibility between the 10j symbol and a boundary structure given by loop quantum gravity-like geometry. Consequently a host of new 4-dimensional models were developed. We here will discuss only a refined version of the original EPR model written with Livine (EPRL) [7], in the case $\gamma < 1$.

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2 Briefly introducing the EPRL model

As is well known, even though GR has local degrees of freedom, it can be put into a 'BF shape' by the use of the action:

$$S_{GR} = \int_M \mathrm{tr}\,(*(e \wedge e) \wedge F(A))$$

for the Lie algebra valued two forms $F(A) = dA + A \wedge A$ and $B = *(e \wedge e) \in \Lambda^2(M, \mathfrak{so}(4))$, where e denotes the (co)frame field $e \in \Lambda^1(M, \mathbb{R}^4)$ (we use the identification $\Lambda^2(\mathbb{R}^4) \simeq \mathfrak{so}(4)$) and $*$ is the $\mathfrak{so}(4)$ Hodge. By restricting the sum over representations and intertwiners in the BF partition function to respect this constraint on the B field, Barrett and Crane derived their 4-dimensional spin foam model [4].

A host of new 4-dimensional models have been recently developed, based on the classically equivalent Holst action:

$$S_{GR} = \int_M \mathrm{tr}\left((*(e \wedge e) + \frac{1}{\gamma} e \wedge e) \wedge F(A)\right)$$

where γ is the so called Immirzi parameter, and the restrictions on the representations and intertwiners are of different form from the BC model. Namely, the EPRL allowed representations are constructed from the Clebsch-Gordan decomposition:

$$V_{j^-} \otimes V_{j^+} \simeq \bigoplus_{k=|j^- - j^+|}^{j^+ + j^-} V_k$$

One then takes the projection onto the highest weight: $k = j^+ + j^-$, and forms the Clebsch-Gordan intertwining map $C_k^{j^- j^+} : V_k \to V_{j^-} \otimes V_{j^+}$ injecting into the highest (diagonal SU(2) subgroup) factor. The labels $j^\pm$ and k are related via the Immirzi parameter for $\gamma < 1$ by

$$j^\pm = \frac{1}{2}(1 \pm \gamma)\, k. \tag{2}$$

This tells us a specific way to go from an SU(2) irrep to a tensor product of two SU(2) irreps, i.e. to a Spin(4) irrep.

Moving on to the intertwiners, an SU(2) intertwiner $\hat{\iota}$ is an element of $\mathrm{Hom}_{\mathrm{SU}(2)}(\mathbb{C}, \bigotimes_{i=1}^4 V_{k_i})$. From the above construction of the injection of irreps of SU(2) into those of Spin(4), and given an SU(2) intertwiner $\hat{\iota}$, a Spin(4) intertwiner ι is constructed as follows:

$$\iota := \int_{\mathrm{Spin}(4)} dG\ (j_i^- \otimes j_i^+)(G) \circ \bigotimes_{i=1}^4 C_{k_i}^{j_i^- j_i^+} \circ \hat{\iota}\ , \tag{3}$$

where the notation $G = (X^-, X^+) \in \mathrm{Spin}(4) = \mathrm{SU}(2) \times \mathrm{SU}(2)$ is used (see figure 1). The group integration ensures that the resulting object is Spin(4)-invariant, i.e., is an element of $\mathrm{Hom}_{\mathrm{Spin}(4)}(\mathbb{C}, \bigotimes_{i=1}^4 V_{(j_i^-, j_i^+)})$.

Labelling the tetrahedra by $a = 1, ..., 5$, the ten triangles Δ_2 of the 4-simplex Δ_4 are then indexed by the pair ab of tetrahedra which intersect on the triangle. There are two SU(2) group elements (X_a^-, X_a^+) and one SU(2) intertwiner $\hat{\iota}_a$ for each tetrahedron. The above Spin(4) intertwiners are glued together in the standard fashion (the usual pentagon combinatorics) to construct an amplitude (a complex number) for this data. Note that now the input data for this 4-simplex Spin(4) amplitude is a spin $k \in \{0, \frac{1}{2}, 1, \ldots\}$ for each triangle of the 4-simplex and an SU(2) intertwiner $\hat{\iota}$ for each tetrahedron.

This concludes the basic construction of the model. We opt for not writing the amplitude in the present form, preferring to first write it into a form appropriate for asymptotic analysis.

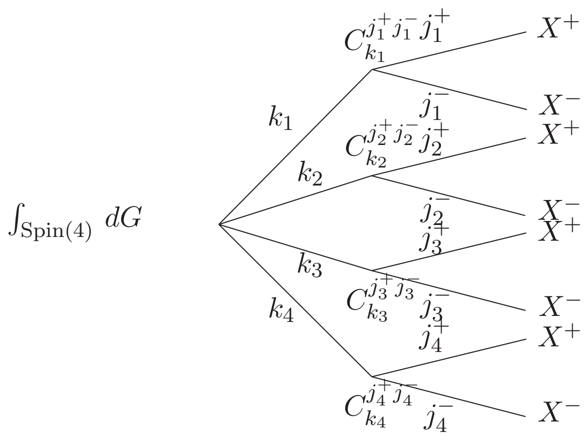

Figure 1: The Spin(4) intertwiner ι.

Nonetheless, one can see that the asymptotic problem could not yet be well posed, because the scaling of the SU(2) intertwiners is not defined in the present form. Solving this problem naturally leads to a reformulation of the integral formula to an exponential form which is particularly well suited to asymptotics.

3 Coherent: states and tetrahedra

States: The fundamental new tool that permitted the asymptotic analysis of the new models was the introduction of *coherent states*. Heuristically, these are states of some irrep of SU(2) that are most geometrical, or semi-classical in the sense that they minimize the uncertainty in total angular momentum [8]. The coherent states have maximal spin projection along the $\mathbf{n}$ axis, i.e. they are highest weight eigenvectors of the normalized Lie algebra elements. Explicitly, for $L^j = \frac{i}{2}\sigma^j$ the Lie algebra generators and $\mathbf{n} \in S^2$, a coherent state $|k, \mathbf{n}\rangle \in V_k$ in direction $\mathbf{n}$ is a unit vector satisfying

$$(\mathbf{L}.\mathbf{n})\, |k, \mathbf{n}\rangle = ik\, |k, \mathbf{n}\rangle \tag{4}$$

where the dot '.' denoting the 3d (Euclidean) scalar product. At each point, there is a U(1) family of coherent states that satisfy (4), and we have denoted a fixed initial arbitrary choice as[1] $|k, \mathbf{n}\rangle$.

Coherent states have the following properties which will be useful to us:

1. $g|k, \mathbf{n}\rangle = e^{ik\phi}|k, \hat{g}\mathbf{n}\rangle$ where $g \in \mathrm{SU}(2)$, with SO(3) projection $\hat{g}$, and ϕ is an arbitrary phase. This means that the action of SU(2) takes a coherent state for a vector $\mathbf{n}$ into a coherent state for a vector $\hat{g}\mathbf{n}$.

2. $|k, \mathbf{n}\rangle = |\frac{1}{2}, \mathbf{n}\rangle^{\otimes 2k} =: |\mathbf{n}\rangle^{\otimes 2k}$ so coherent states exponentiate into the fundamental representation, which in diagrammatical calculus means we replace a strand labelled k by $2k$ *identical* fundamental strands.

Tetrahedra: To construct a coherent intertwiner associated to a tetrahedron, the idea is to associate a coherent state to each one of its triangles and then integrate over SU(2). The geometrical picture is that the coherent intertwiner corresponding to tetrahedron a of the triangulation will be given by a 'coherent tetrahedron' labeled τ_a. Here τ_a has a coherent state $|k_{ab}, \mathbf{n}_{ab}\rangle$ for each face,

[1] For fixed k this is equivalent to a section of the Hopf bundle, $s : S^2 \simeq \mathrm{SU}(2)/\mathrm{U}(1) \to \mathrm{SU}(2)$. Locally we can denote any other choice by $e^{i\theta(\mathbf{n})}|k, \mathbf{n}\rangle$.

carrying the interpretation of the normals of length k and direction $\mathbf{n}_{ab}$ (and an implicit choice of phase factor). Thus apart from the phase factor, we can in effect regard τ_a as a tetrahedron in $\mathbb{R}^3$ with the standard metric, with $\mathbf{n}_{ab}$ and k_{ab} being the normal and area associated to $\tau_{ab} \subset \tau_a$; the triangle of τ_a (combinatorically) adjacent to tetrahedron b.

Of course, we want to describe tetrahedra with three-dimensional rotational symmetry, so the coherent intertwiners are constructed by integrating over all spatial directions the tensor product of four coherent states

$$\hat{\imath}(\mathbf{n}_1, \mathbf{n}_2, \mathbf{n}_3, \mathbf{n}_4) = \int_{\mathrm{SU}(2)} dh \bigotimes_{i=1}^{4} h|k_i, \mathbf{n}_i\rangle. \tag{5}$$

These intertwiners were introduced by Livine and Speziale [8], who gave an asymptotic formula for their normalisation.

According to the 'quantization commutes with reduction' theorem of Guillemin and Sternberg [9], the space of intertwiners is spanned by the $\hat{\imath}$ determined by vectors satisfying the *closure constraint* $k_1\mathbf{n}_1 + k_2\mathbf{n}_2 + k_3\mathbf{n}_3 + k_4\mathbf{n}_4 = 0$. Thus we take the coherent intertwiners to always satisfy this condition and thus be given by some tetrahedron τ.

Given the above formulation of the coherent intertwiner for SU(2), we know from equation (3) what the form of the Spin(4)-intertwiner should be.[2]

4 Exponential form and stationary points

Writing the amplitude in exponential form The amplitude $f_4 \in \mathbb{C}$ is defined by forming a closed spin network diagram from the 5 Spin(4) intertwiners (vertices) ι_a, which are tensored together and then the free ends are joined pairwise according to the combinatorics. This is done using the standard 'ϵ inner product' of irreducible representations of SU(2), denoted $\epsilon_k \colon V_k \otimes V_k \to \mathbb{C}$. This inner product is represented in the spin network diagram as a semicircular arc[3]. To toggle between the usual Hermitian inner product and the epsilon inner product, one uses the standard antilinear structure map for representations of SU(2), $J \colon V_k \to V_k$. This is defined by

$$\epsilon_k(\alpha, \alpha') = \langle J\,\alpha | \alpha' \rangle,$$

the left-hand side being the epsilon-inner product and the right hand side the Hermitian inner product. It obeys $Jg = gJ$ for all $g \in \mathrm{SU}(2)$, $J^2 = (-1)^{2k}$ and $\langle J\alpha | J\alpha' \rangle = \overline{\langle \alpha | \alpha' \rangle}$. Furthermore, since $J(i\mathbf{n} \cdot \mathbf{L}) = -(i\mathbf{n} \cdot \mathbf{L})J$, the map J takes a coherent state for $\mathbf{n}$ to a coherent state for $-\mathbf{n}$, hence the notation $|k_{ab}, -\mathbf{n}_{ab}\rangle$ means $J|k_{ab}, \mathbf{n}_{ab}\rangle$.

To combine the Spin(4) intertwiners ι_a, one first of all regards each vertex as an SU(2) spin network (as in figure 1), and uses one ϵ inner product to connect the j^+ edges and a second ϵ inner product to connect the j^- edges. Using the form (5) for the intertwiners, one splits the total amplitude into a $\mathrm{Spin}(4)^5$ integral, where the integrand is a product of spin network evaluations, one for each edge of the 4-simplex (see also footnote 2). We call these evaluations *propagators*, denoted by $\mathcal{P}_{ab}$.

It is easy to see that the symmetrizers on the j^+_{ab} and j^-_{ab} edges can be absorbed into the symmetrizer on the k_{ab} edge because of the stacking property of symmetrizers. Furthermore, using the exponentiating property of the coherent states, the remaining symmetrizer now acts redundantly on the coherent states $|k_{ab}, \mathbf{n}_{ab}\rangle$, and we can further split the propagator into the

[2] Note that the invariance of the Clebsch-Gordan injections permits us to absorb the SU(2) integration of the coherent tetrahedra into the Spin(4) integration.

[3] This is defined by a choice of the two-dimensional antisymmetric tensor ϵ for SU(2) spin 1/2, and extended to arbitrary spin by tensor products of ϵ. This choice of inner product makes the combinatorics and -1 signs tractable, but also allows the natural assignment of coherent states (and hence intertwiners) to the triangle normals.

j^+ and j^- strands, i.e. to a product of terms in the fundamental representation. We obtain the following expression for the propagator

$$\mathcal{P}_{ab} = \langle -\mathbf{n}_{ab}|(X_a^-)^{-1}X_b^-|\mathbf{n}_{ba}\rangle^{2j_{ab}^-}\,\langle -\mathbf{n}_{ab}|(X_a^+)^{-1}X_b^+|\mathbf{n}_{ba}\rangle^{2j_{ab}^+}. \tag{6}$$

The four-simplex amplitude can thus be re-expressed as $f_4 = \int_{\mathrm{Spin}(4)^5}\prod_a dG_a\, e^S$ with the action given by

$$S = \sum_{a<b} 2j_{ab}^- \ln\langle -\mathbf{n}_{ab}|(X_a^-)^{-1}X_b^-|\mathbf{n}_{ba}\rangle + 2j_{ab}^+ \ln\langle -\mathbf{n}_{ab}|(X_a^+)^{-1}X_b^+|\mathbf{n}_{ba}\rangle. \tag{7}$$

Stationary Points We start by scaling all ten spins by a constant parameter $k_{ab} \to \lambda k_{ab}$. Our strategy is to use extended stationary phase methods, that is, stationary phase generalized to (non purely imaginary) complex functions, to find, in terms of the boundary data, the Spin(4) elements $G_a = (X_a^-, X_a^+)$ that leave the action stationary.

In the extended stationary phase, the key role is played by *critical points*, i.e stationary points for which Re$S = 0$. If S has no critical points then for large parameter λ the function f decreases faster than any power of λ^{-1}. In other words, for all $N \geq 1$: $f(\lambda) = o(\lambda^{-N})$. Otherwise, for large λ the asymptotic expansion of the integral $f(\lambda) = \int_D dx\, a(x)\, e^{\lambda S(x)}$ yields for each critical point [10][4]

$$a(x_0)\left(\frac{2\pi}{\lambda}\right)^{n/2}\frac{1}{\sqrt{\det(-H)}}\,e^{\lambda S(x_0)}\left[1 + O(1/\lambda)\right].$$

where H denotes the Hessian matrix of S. The real part of the action (7) is given by

$$\mathrm{Re}\, S = \sum_{a<b} j_{ab}^- \ln\frac{1}{2}(1 - \mathbf{n}_{ab}^- \cdot \mathbf{n}_{ba}^-) + j_{ab}^+ \ln\frac{1}{2}(1 - \mathbf{n}_{ab}^+ \cdot \mathbf{n}_{ba}^+), \tag{8}$$

where $\mathbf{n}_{ab}^\pm := X_a^\pm \mathbf{n}_{ab}$ and we have used the expression of the inner product between coherent states and all phases have been absorbed in the imaginary part of the action. The maximality equation and the critical equation (obtained by using standard SU(2) coherent state identities on the first variation formula), become respectively:

$$X_a^\pm \mathbf{n}_{ab} = -X_b^\pm \mathbf{n}_{ba} \tag{9}$$

$$\sum_{b:\, b\neq a} k_{ab}\,\mathbf{n}_{ab} = 0 \tag{10}$$

for all $a = 1, ..., 5$. The second one, implying closure of the coherent tetrahedra, is redundant, since we already chose our states to be of this form.

5 Bivectors, Gluing, and Boundary data

Bivectors and Gluing Now that we have the stationarity equations, the programme is to input them back into the action and give them a geometric interpretation, à la Ponzano-Regge. The first obstacle is that these equations involve basically 3-dimensional rotations acting on vectors, but we would like to give them an interpretation of 4-dimensional geometry.

What one does first is to regard the coherent tetrahedra as immersed in $\mathbb{R}^4$. Let us say lying in the plane $x_0 = 0$, i.e. in the plane orthogonal to $\mathbf{e}_0 = (1, 0, 0, 0)$. Now, we immerse the vectors normal to the associated triangles of the tetrahedron, $k_{ab}\mathbf{n}_{ab} \in \mathbb{R}^3$, by canonically associating them to bivectors $B_0(k_{ab}\mathbf{n}_{ab}) = (b_0^-(\mathbf{n}_{ab}), b_0^+(\mathbf{n}_{ab}))$ given in the '(self-dual, anti-self-dual)' decomposition.

[4]The stationary points are assumed to be isolated and non-degenerate; $\det H \neq 0$

Namely, we map $k_{ab}\mathbf{n_{ab}} \mapsto k_{ab}(\mathbf{n_{ab}}, \mathbf{n_{ab}})$, which is a simple bivector and still lies in $\mathbf{e}_0^\perp$. By acting on this bivector with $G = (X_a^-, X_a^+)$ we get

$$B_{ab} := k_{ab}\,(X_a^-, X_a^+)(\mathbf{n}_{ab}, \mathbf{n}_{ab}). \tag{11}$$

which now lies in the plane orthogonal to $G_a\mathbf{e}_0$, and is still simple, since $|b^+| = |b^-|$. Since for all $b \neq a$, the B_{ab} lie in the same hyperplane, $(G_a\mathbf{e}_0)^\perp$, it can be shown that the set of bivectors satisfy the so called 'cross-simplicity constraints' [8] as well.

It can furthermore easily be shown that these bivectors satisfy all but one of the 'bivector geometry conditions'. By the *reconstruction theorem* in [4], the full set are the conditions that allows the set of bivectors to determine a unique non-degenerate geometric 4-simplex σ in $\mathbb{R}^4$ (defined up to translation and inversion). The condition not yet satisfied is non-degeneracy, in the sense that for six triangles sharing a common vertex we do not know if the six bivectors are linearly independent.

To address this, we must go back to the stationarity equations, (10) and (9). Given a set $\mathcal{B} = \{\mathbf{n}_{ab}, k_{ab}\}_{a\neq b}$ of boundary data (with phases of coherent states still undetermined) satisfying (10), suppose there exists two sets of five SU(2) elements $\{U_a^+\}$ and $\{U_a^-\}$ which solve $U_a\mathbf{n}_{ab} = -U_b\mathbf{n}_{ba}$. Suppose furthermore that the solutions are distinct (not related by a global symmetry) $\{U_a^+\} \sim \{U_a^-\}$. Then, equating $\{X_a^-, X_a^+\} = \{U_a^-, U_a^+\}$ it is straightforward to prove that indeed the bivectors defined in (11) are non-degenerate (see Lemma 3 of [1]). Hence, by the reconstruction theorem of bivectors, they determine a unique geometric 4-simplex; the unique one (up to translation and inversion) [5] compatible with the boundary tetrahedra given by $\{\mathbf{n}_{ab}, k_{ab}\}_{a\neq b}$.

It then follows that we can have at most two distinct sets of solutions $\{U_a\}$, since, were there a third set, we would, by the reconstruction theorem, be able to generate a distinct geometric 4-simplex compatible with the same boundary tetrahedra.

Boundary data Let us *focus on the case where the boundary data allows two solutions* $\{U_a^-, U_a^+\}$. Then, by the reconstruction theorem we have that the geometry and orientation of the triangles τ_{ab} and τ_{ba} (associated to the coherent tetrahedra τ_a and τ_b resp.) must be a priori compatible. Therefore we have that there exists a unique $\hat{g}_{ab} \in \mathrm{SO}(3)$ for which we have

$$\begin{aligned} \hat{g}_{ab}(\tau_{ab}) &= \tau_{ba} \\ \hat{g}_{ab}\mathbf{n}_{ab} &= -\mathbf{n}_{ba}. \end{aligned} \tag{12}$$

This is called Regge-like boundary in [1].

Furthermore, in the discussion on coherent states we saw that we were left with an arbitrary U(1) rotation to determine. The choice of phase for the boundary state above is given by picking the phase $|k_{ab}, \mathbf{n}_{ab}\rangle_{\mathrm{R}}$ for τ_a to be arbitrary, and then fixing the phase of the state for the corresponding triangle in τ_b to be

$$|k_{ab}, \mathbf{n}_{ba}\rangle_{\mathrm{R}} = g_{ab}J|k_{ab}, \mathbf{n}_{ab}\rangle_{\mathrm{R}}. \tag{13}$$

we denote this choice by the sub-index R (or Regge). Regge-like boundary data together with this choice of phase for the boundary state is called a *Regge state*.

6 Dihedral angles and asymptotic formula

The idea here is that the geometrically induced choice of phases will correlate the value of the action at the critical points with the dihedral angles.

[5] It can be shown that equating instead $\{X_a^\pm\} = \{U_a^\mp\}$ yields the 4-simplex with opposite orientation.

Dihedral angles Suppose N_a is the outward unit normal vector to tetrahedron a. Then $N_a \wedge N_b$ defines a bivector which is in the plane orthogonal to the triangle where tetrahedra a and b intersect. Therefore $*(N_a \wedge N_b)$ lies in the plane of the triangle; normalising it correctly then equals the definition of the bivector B_{ab} (11). The dihedral rotation $\widehat{D}_{ab} \in \mathrm{SO}(4)$ is defined as the rotation that maps the normal $N_a = G_a \mathbf{e}_0$ to the normal N_b and stabilizes the orthogonal plane $N_a^\perp \cap N_b^\perp$ (the plane of the bivector B_{ab}). These comments permit us to write[6]:

$$
\begin{aligned}
D_{ab} &:= \exp\left(\Theta_{ab} \frac{N_b \wedge N_a}{|N_b \wedge N_a|}\right) && (14)\\
B_{ab} &= k_{ab} * \frac{N_a \wedge N_b}{|N_a \wedge N_b|} = k_{ab}(X_a^-, X_a^+)(\mathbf{n}_{ab}, \mathbf{n}_{ab}) && (15)
\end{aligned}
$$

Acting with the Hodge on (15) and using $** = 1$ and (14) leads to

$$
D_{ab} = \left(\exp\left(-\Theta_{ab}(\hat{X}_a^- \mathbf{n}_{ab}) \cdot \mathbf{L}\right), \exp\left(\Theta_{ab}(\hat{X}_a^+ \mathbf{n}_{ab}) \cdot \mathbf{L}\right)\right). \quad (16)
$$

Now consider the following diagram:

$$
\begin{tikzcd}
\tau_a \arrow[r, "(X_a^-, X_a^+)"] \arrow[d, "(g_{ab}, g_{ab})"'] & \sigma_a \arrow[d, "D_{ab}"] \\
\tau_b \arrow[r, "(X_b^-, X_b^+)"'] & \sigma_b
\end{tikzcd} \quad (17)
$$

where $\tau_a \subset \mathbb{R}^4$ are the tetrahedra at $\mathbf{e}_0^\perp$, and $\sigma_a \in \mathbb{R}^4$ are the actual geometrical ones in the 4-simplex σ and the $g_{ab} \in \mathrm{SU}(2)$, are defined in (12). Note that by the reconstruction theorem, the maps in the diagram commutes when acting on both the triangles τ_{ab} and on the internal normals $\mathbf{n}_{ab}$, hence (since all maps in the diagram are orientation preserving) the SO(4) action of the maps in the diagram commutes. Thus acting with $\left((X_a^-)^{-1}, (X_a^+)^{-1}\right)$ on to the left of the commuting diagram equation one gets the two equations:

$$
(X_a^\pm)^{-1} X_b^\pm g_{ab} \quad = \quad \exp\left(\mp i \Theta_{ab} \mathbf{n}_{ab} \cdot L\right) \quad (18)
$$

Now, as we know, the critical points satisfy closure and the conditions $(X_a^\pm)^{-1} X_b^\pm (\mathbf{n}_{ba}) = -\mathbf{n}_{ab}$, for all $a \neq b$. The lift of this equation to the coherent states involves a phase

$$
(X_a^\pm)^{-1} X_b^\pm |\mathbf{n}_{ba}\rangle = e^{i\phi_{ab}^\pm} |-\mathbf{n}_{ab}\rangle. \quad (19)
$$

Then, taking the inner product with $\langle -\mathbf{n}_{ab}|$, with the Regge phase choice (paying special attention to the indices and signs):

$$
\begin{aligned}
e^{i\phi_{ab}^\pm} &= {}_{\mathrm{R}}\langle -\mathbf{n}_{ab}|(X_a^\pm)^{-1} X_b^\pm |\mathbf{n}_{ba}\rangle_{\mathrm{R}} && (20)\\
&= {}_{\mathrm{R}}\langle -\mathbf{n}_{ab}|(X_a^\pm)^{-1} X_b^\pm g_{ab}| -\mathbf{n}_{ab}\rangle_{\mathrm{R}} = \overline{\langle \mathbf{n}_{ab}| \exp\left(\mp \Theta_{ab} \mathbf{n}_{ab} \cdot L\right) |\mathbf{n}_{ab}\rangle_{\mathrm{R}}} && (21)\\
&= e^{\pm \frac{i}{2} \Theta_{ab}} && (22)
\end{aligned}
$$

where in the first equality of (21) we have used the Regge phase choice (13), and in the second equality of the same line we have used (18) and the properties of the J map.

Finally, by a simple direct computation, (7) becomes:

$$
S = \gamma \sum_{a<b} k_{ab} \Theta_{ab}. \quad (23)
$$

[6]A bivector $N \wedge M = N \otimes M - M \otimes N$, as an element of the Lie algebra $\mathfrak{so}(4)$, acts on vectors through the Euclidean metric inner product. Also note that the isomorphism $\mathbb{R}^3 \to \mathfrak{su}(2)$ is effected through $v \mapsto v \cdot \mathbf{L}$.

Even for a boundary state $\{|k_{ab}, \mathbf{n}_{ab}\rangle_{\mathrm{R}}\}_{a\neq b}$ such that $\{U_a^+\} \nsim \{U_a^-\}$, we may still form degenerate solutions of the form $\{X_a^\pm\} \sim \{U_a^+\}$ or $\{X_a^\pm\} \sim \{U_a^-\}$. These will contribute with the strength $\pm\sum_{a<b} k_{ab}\Theta_{ab}$ to the asymptotic formula.

Thus for non-degenerate boundary data we write for the total amplitude from (4) and (7):

$$f_4(\{\lambda k_{ab}, |\mathbf{n}_{ba}\rangle_{\mathrm{R}}\}) \simeq \left(\frac{2\pi}{\lambda}\right)^{12} \left[2N^\gamma_{+-}\cos\left(\lambda\gamma\sum_{a<b} k_{ab}\Theta_{ab}\right) + N^\gamma_{++}\exp\left(i\lambda\sum_{a<b} k_{ab}\Theta_{ab}\right) + N^\gamma_{--}\exp\left(-i\lambda\sum_{a<b} k_{ab}\Theta_{ab}\right)\right] \tag{24}$$

where the N's are prefactors depending on the determinant of the Hessian but not on λ. As in the case of the Ponzano-Regge model, a cosine term appears because simplex geometries with either of the two possible orientations can occur.

7 Conclusion

We have studied the semi-classical limit of the four-simplex amplitude of the Euclidean EPRL model for $\gamma < 1$ and maximal set of solutions for the stationarity equations. The asymptotic formula contains the cosine of the Regge action. However our asymptotic formula also contains two additional terms, with exponentials of the same Regge action formula, but without the Immirzi parameter γ. Interestingly, in the asymptotic limit, all of these terms scale with the same exponent of the asymptotic parameter λ.

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